[1]This mini note comes from [1].

The *Regge symmetry* R is a linear map on $\mathbb{R}^6$:

$R : (x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}) \to (x_{12}, x - x_{13}, x - x_{14}, x - x_{23}, x - x_{24}, x_{34}),$

where

$$x = \frac{1}{2}(x_{13} + x_{14} + x_{23} + x_{24}).$$

Two Euclidean polyhedron are said to be *scissors congruent* if one can be cut into pieces and re-assembled to give the other.

We will show how Regge symmetry produce two scissor congruent tetrahedra.

Label the vertices of a Euclidean tetrahedron T by 1,2,3,4. Let the length of the edge joining vertices i, j be l_{ij}.

Theorem 1 (Ponzano-Regge 1968)**.** *The Euclidean tetrahedron T' with lengths $R(l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34})$ has the same volume as T.*

This is proved by applying the following Cayley-Menger formula which calculate the volume $V(T)$ of the tetrahedron T:

$$V(T)^2 = \frac{(-1)^{3+1}}{2^3(3!)^2} \det \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & l_{12}^2 & l_{13}^2 & l_{14}^2 \\ 1 & l_{12}^2 & 0 & l_{23}^2 & l_{24}^2 \\ 1 & l_{13}^2 & l_{23}^2 & 0 & l_{34}^2 \\ 1 & l_{14}^2 & l_{24}^2 & l_{34}^2 & 0 \end{pmatrix}.$$

Let θ_{ij} and θ'_{ij} be the dihedral angles of the tetrahedron T and T' respectively.

Theorem 2 (Ponzano-Regge 1968)**.**

$$(\theta'_{12}, \theta'_{13}, \theta'_{14}, \theta'_{23}, \theta'_{24}, \theta'_{34}) = R(\theta_{12}, \theta_{13}, \theta_{14}, \theta_{23}, \theta_{24}, \theta_{34}).$$

This is proved by a tour-de-brute-force of trigonometry.

Define the *Dehn invariant* of the tetrahedron T to be

$$\delta(T) = \Sigma_{i,j} l_{ij} \otimes \theta_{ij}.$$

Corollary 3. *T and T' have the same Dehn invariants.*

In 1965, Sydler showed that two polyhedra are scissors congruent if and only if they have the same volume and the same Dehn invariants. Thus tetrahedra T, T' are scissor congruent. The question is can you construct this scissor congruence without using this result.

Remark 4. Regge symmetry was discovered by calculation of $6j$ symbol. In fact, $6j$ symbol is invariant under Regge symmetry.

Remark 5. Mohanty [2] proved Regge symmetry (defined by using dihedral angles) can produce two scissors congruent hyperbolic tetrahedra. The proof is constructive. But it does not work for the Euclidean case.

References

1. Justin Roberts, *Classical 6j-symbols and the tetrahedron*, Geom. Topol. 3 (1999), 21-66
2. Yana Mohanty, *The Regge symmetry is a scissors congruence in hyperbolic space*, Algebr. Geom. Topol. 3 (2003) 1-31

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