[1]This is a note of section 10.1 in the book *A Course in Enumeration* by Martin Aigner, GTM 238, Springer, 2007

A *perfect matching* of a graph is a set of disjoint edges covering all of the vertices. In statistical mechanics, a perfect matching is called a dimer configuration of a dimer model. Assume that a plane graph G has no loops, any two vertices are connected by at most one edge and the degree of each vertex is at least two. The goal is to count the number of perfect matchings of a graph G.

We can orient the edges of G such that the number of clockwise oriented edges of each face of G is odd (it is called the $\mathcal{O}$ orientation). This can be proved by induction. Indeed, it is true for a graph with one face. By deleting an edge, one can reduce the number of faces of a graph.

Denote by $c(C)$ the number of clockwise oriented edges of each *circuite* C (a closed path formed by edges) and by $p(C)$ the number of vertices in the interior of the region bounded by C. If a graph G is equipped the $\mathcal{O}$ orientation, then

$$c(C) + p(C) \equiv 1 \mod 2. \tag{1}$$

This is true if C bounds a face of G (or $p(C) = 0$). For general circuite, it is proved by applying Euler's formula concerning number of vertices, edges and faces.

For any oriented graph G, its *signed adjacency matrix* $A = (a_{ij})$ is defined by $a_{ij} = 1$ if $i \to j$, $a_{ij} = -1$ if $j \to i$ and $a_{ij} = 0$ if i, j are not jointed by an edge.

Let $2n$ be the number of vertices of G. Let μ be a pairing of vertices of G, i.e., $\mu = i_1j_1, i_2j_2, ..., i_nj_n$ is a paring of the set $\{1, 2, ..., 2n\}$. Denote by sign μ the sign of the permutation $\begin{pmatrix} 1 & 2 & ... & 2n-1 & 2n \\ i_1 & j_1 & ... & i_n & j_n \end{pmatrix}$. Let $P(2n) = \{\mu\}$ be the set of pairings.

Let $S_e(2n)$ be the set of permutations which can be written as a product of cycles with even length. We claim there is a bijection $P(2n) \times P(2n) \to S_e(2n)$. It is explained by the following example.

For example, if $\mu_1 = 14, 28, 35, 67$ and $\mu_2 = 15, 26, 34, 78$, represent μ_1 by arcs above the x-axis connecting numbers 1 and 4, 2 and 8, etc. Represent μ_2 by arcs below the x-axis connecting numbers 1 and 5, 2 and 6, etc. Then $(\mu_1, \mu_2) \to \sigma = (1435)(2876)$ which is corresponding to the cycles formed by the arcs, where each cycle of σ is written in the canonical way: smallest element first.

For the signed adjacency matrix $A = (a_{ij})$ (generically, a anti-symmetric matrix), denote $a_\mu = a_{i_1j_i}...a_{i_nj_n}$ for a paring $\mu = i_1j_1, i_2j_2, ..., i_nj_n$. And denote $a_\sigma = \Pi_i a_{i\sigma(i)}$ for a permutation σ. If $(\mu_1, \mu_2) \to \sigma$, we can check that

$$(\text{sign } \mu_1)a_{\mu_1} \cdot (\text{sign } \mu_2)a_{\mu_2} = (\text{sign } \sigma)a_\sigma. \tag{2}$$

This property is used to prove Cayley's theorem that $\det(A) = [\text{Pf}(A)]^2$ for an anti-symmetric $2n \times 2n$ matrix A, where the *Pfaffian* of A is defined as

$$\text{Pf}(A) = \sum_{\mu \in P(2n)} (\text{sign } \mu)a_\mu. \tag{3}$$

For a graph G with signed adjacency matrix A, if G is equipped the $\mathcal{O}$ orientation, by the property (1), we see that $(\text{sign } \sigma)a_\sigma = 0$ or 1 for any $\sigma \in S_e(2n)$. This works only for plane graph. Thus by property (2), $(\text{sign } \mu)a_\mu = \pm 1$ have the same sign for all $\mu \in P(2n)$. Therefore by (3), we have $\sqrt{\det(A)} = |\text{Pf}(A)| = \sum_{\mu \in P(2n)} |a_\mu|$ is the number of perfect matchings of the graph G.

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