[1]This mini note stems from the reading of section 3 of [1].

Given a polygon with $2n$ edges, a choice of an orientation of the boundary circle gives an orientation of each edge. There are $(2n-1)!! = 1 \cdot 3 \cdots (2n-1)$ ways to pair these $2n$ edges. Gluing each pair of edges in the opposite directions produces an oriented surface. The question is to count the number $\varepsilon_g(n)$ of surfaces of genus g in these resulting surfaces.

Let's introduce the generating function:

$$T_n(N) := \sum_{\sigma} N^{V(\sigma)} = \sum_{g=0}^{\infty} \varepsilon_g(n) N^{n+1-2g},$$

where the first sum is taken over $(2n-1)!!$ ways of paring $\{\sigma\}$ and $V(\sigma)$ is the number of vertices in the resulting surface corresponding to paring σ. The second equality is due to rearranging the term according the power and Euler's formula.

Define the Gaussian measure on $\mathbb{R}^k$ to be

$$d\mu(x) = (2\pi)^{-\frac{k}{2}} (\det B)^{\frac{1}{2}} \exp(-\frac{1}{2}(Bx, x)) dx_1 \cdots dx_k,$$

where B is a positive definite symmetric $k \times k$ matrix. We should be able to check $\int_{\mathbb{R}^k} d\mu(x) = 1$.

For any function $f : \mathbb{R}^k \to \mathbb{C}$, we define the mean of f by $\langle f \rangle = \int_{\mathbb{R}^k} f(x) d\mu(x)$.

Theorem 1 (Wick formula)**.** *Let $f_1, f_2, ..., f_{2n}$ be linear functions of $x_1, x_2, ..., x_k$, then*

$$\langle f_1 f_2 \cdots f_{2n} \rangle = \sum \langle f_{p_1} f_{q_1} \rangle \langle f_{p_2} f_{q_2} \rangle \cdots \langle f_{p_n} f_{q_n} \rangle$$

where the sum is taken over all pairing of $\{1, 2, ..., 2n\}$.

Let $H = (h_{ij})$ be an Hermitian $N \times N$ matrix, i.e., $h_{ij} = \overline{h}_{ji} \in \mathbb{C}$. We denote $\mathcal{H}_N$ the space of all such matrixes. Thus $\mathcal{H}_N \cong \mathbb{R}^{N^2}$ with coordinates $h_{ii} \in \mathbb{R}, i = 1, ..., N$ and $\Re(h_{ij}), \Im(h_{ij}), 1 \le i < j \le N$.

Let's introduce a special Gaussian measure on $\mathcal{H}_N \cong \mathbb{R}^{N^2}$ by

$$d\mu(H) = (2\pi)^{-\frac{N^2}{2}} 2^{\frac{(N^2-N)}{2}} \exp(-\frac{1}{2} tr(H^2)) \prod_{i=1}^{N} h_{ii} \prod_{i<j} d\Re(h_{ij}) d\Im(h_{ij}).$$

Lemma 2. *$\langle h_{ij} h_{ji} \rangle = 1$ and $\langle h_{ij} h_{kl} \rangle = 0$ whenever $(ij) \neq (kl)$.*

It is proved by simple calculation.

Theorem 3. $\int_{\mathcal{H}_N} tr(H^{2n}) d\mu(H) = T_n(N)$.

It is a corollary of Theorem 1 and Lemma 2. After working on the example $n = 2$, we can see why the theorem should be true. Notice $tr(H^4) = \sum_{i,j,k,l=1}^{N} h_{ij} h_{jk} h_{kl} h_{li}$. By Wick formula

$\langle h_{ij} h_{jk} h_{kl} h_{li} \rangle = \langle h_{ij} h_{jk} \rangle \langle h_{kl} h_{li} \rangle + \langle h_{ij} h_{kl} \rangle \langle h_{jk} h_{li} \rangle + \langle h_{ij} h_{li} \rangle \langle h_{jk} h_{kl} \rangle$. By Lemma 2, the first (or third) term is equal to 1 iff $i = k$ (or $j = l$). Thus in $\langle tr(H^4) \rangle$, the contribution of the first (or third) term is N^3. The second term is equal to 1 iff $i = j = k = l$. Thus in $\langle tr(H^4) \rangle$, the contribution of the second term is N.

References

1. S.K.Lando & A.K.Zvonkin, *Graphs on Surfaces and Their Application*, EMS Vol.141, Springer, 2004

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[1]November 6, 2007, $\mathcal{SQPS}$